New Algebraic Quantum Many-body Problems*

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March 1, 2000

Abstract

We develop a systematic procedure for constructing quantum many-body problems whose spectrum can be partially or totally computed by purely algebraic means. The exactly-solvable models include rational and hyperbolic potentials related to root systems, in some cases with an additional external field. The quasi-exactly solvable models can be considered as deformations of the previous ones which share their algebraic character.

Submitted to *J. Phys. A*

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1 Introduction

Since the pioneering works of Calogero and Sutherland, [1, 2, 3], much effort has been invested in the study of quantum many-body problems. A thorough classification of completely integrable Hamiltonians related to root systems has been performed in the early eighties by Olshanetsky and Perelomov, both in the classical, [4], and the quantum cases, [5]. The complete integrability of these models is associated to an underlying root system structure, the integrals of motion being related to the radial parts of the Laplace–Beltrami operator on a symmetric space with the given root system.

Some years ago, Turbiner showed using Lie-algebraic techniques that many (though not all) of Olshanetsky–Perelomov's (OP) Hamiltonians are also exactly solvable, [6, 7]. By exact solvability we mean here that the N-body Hamiltonian preserves an infinite increasing sequence of subspaces of known smooth functions, whereas a quasi-exactly solvable Hamiltonian just preserves a single finite-dimensional subspace. The construction of these invariant subspaces is usually based on the theory of representations of Lie algebras of differential operators, [8].

The idea of constructing exactly solvable many-body problems based on the zeros and poles of special solutions of partial differential equations has long been known. The original work [9] explores in depth this relation at the classical level. This idea has been exploited by Hou and Shifman in a recent paper, [10], which extends this approach to the quantum case using Lie-algebraic techniques, thereby obtaining a new family of quasi-exactly solvable many-body potentials.

In this paper we apply Calogero's procedure for constructing solvable many-body problems to the most general quasi-exactly solvable operator on the line admitting square-integrable eigenfunctions, finding several families of exactly and quasi-exactly solvable many-body problems on the line. The exactly solvable Hamiltonians include Calogero's original model and many of Olshanetsky–Perelomov's non-periodic Hamiltonians, and also a family of potentials which are not directly related to a root system due to the presence of an external field. The latter model has been studied by Inozemtsev and Meshcheryakov, [11], who derived its discrete spectrum from that of the standard hyperbolic BC_N model by a limiting process.

It should be emphasized that, although the construction presented here leads to many previously known potentials based on root systems, our approach is entirely different from the usual one, in that no use needs to be made of any underlying root system structure. The main advantage of this algebraic method over the root system approach is that, while the latter is rather rigid, the former's flexibility allows for deformations of the exactly solvable models preserving to some extent their integrability properties.

The paper is organized as follows. In Section 2 some basic definitions are given, and the construction of the general many-body Hamiltonian is explained. The algebraization of this Hamiltonian is discussed in Section 3 for the five families of normalizable quasi-exactly solvable operators on the line. In Section 4 the energy spectrum of all exactly-solvable models obtained in the previous Section eis given explicitly. In Section 5 we work out a few concrete examples, and we sum up the conclusions and outline future work in Section 6. Some useful expressions for handling symmetric variables can be found in the

Appendix.

2 Derivation of H_N

A Schrödinger operator

$$H = -\sum_k \partial_{x_k}^2 + V(\mathbf{x}) \tag{1}$$

(where $\mathbf{x}$ belongs to an open subset of Euclidean space) is said to be *quasi-exactly solvable* (QES) if it preserves a known finite-dimensional subspace $\mathcal{M}$ of smooth functions. The spectral problem for H reduces in this space to diagonalizing the matrix of $H|_{\mathcal{M}}$, which makes it possible to compute a finite subset of the spectrum of H by purely algebraic means. If H preserves an infinite increasing sequence of known finite-dimensional subspaces $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \dots$ then we shall call it *exactly solvable* (ES), since in this case one can algebraically compute an arbitrary number of eigenfunctions and eigenvalues of H by restricting it to each $\mathcal{M}_k$. (If $\mathcal{M}$ or the $\mathcal{M}_k$'s are not in L^2, some of the eigenfunctions of H computed in this way may turn out not to be square-integrable; see [12] for an analysis of this issue in the one-dimensional case.)

The QES (or ES) character of a Schrödinger operator is invariant under a natural group of equivalence transformations, generated by changes of coordinates $\mathbf{x} \mapsto \mathbf{z} = \mathbf{Z}(\mathbf{x})$ and conjugation by arbitrary non-negative functions $\mu(\mathbf{z})$, which map $H \equiv H(\mathbf{x})$ into the differential operator (not necessarily of Schrödinger type)

$$\overline{H}(\mathbf{z}) = \mu(\mathbf{z})^{-1} H(\mathbf{x})\, \mu(\mathbf{z})\,. \tag{2}$$

Thus, if $H(\mathbf{x})$ preserves $\mathcal{M} \equiv \mathcal{M}(\mathbf{x})$ then $\overline{H}(\mathbf{z})$ preserves the subspace

$$\overline{\mathcal{M}}(\mathbf{z}) = \mu(\mathbf{z})^{-1} \mathcal{M}(\mathbf{x})\,, \tag{3}$$

and if $\psi(\mathbf{x})$ is an eigenfunction of $H(\mathbf{x})$ with eigenvalue E belonging to $\mathcal{M}(\mathbf{x})$ then $\overline{\psi}(\mathbf{z}) = \mu(\mathbf{z})^{-1}\psi(\mathbf{x})$ is an eigenfunction of $\overline{H}(\mathbf{z})$ lying in $\overline{\mathcal{M}}(\mathbf{z})$ with the same energy E. Note, however, that such an equivalence transformation may not preserve the square integrability of the eigenfunctions, since $\mu(\mathbf{z})$ is not required to be unimodular.

A very general way of constructing QES Schrödinger operators is to start with a finite-dimensional Lie algebra of differential operators $\bar{\mathfrak{g}}$ admitting finite-dimensional representations in the space of smooth functions in the variable $\mathbf{z}$ (usually called a *quasi-exactly solvable algebra* in the literature). Any differential operator $\overline{H}(\mathbf{z})$ belonging to the enveloping algebra of $\bar{\mathfrak{g}}$ will automatically preserve the carrier space $\overline{\mathcal{M}}(\mathbf{z})$ of such a finite-dimensional representation. If one can find an equivalence transformation (2) such that the differential operator $H(\mathbf{x})$ is of Schrödinger type (1), then $H(\mathbf{x})$ is clearly a QES Schrödinger operator, since it preserves the finite-dimensional subspace $\mathcal{M}(\mathbf{x}) = \mu(\mathbf{z})\, \overline{\mathcal{M}}(\mathbf{z})$. In this case, one says that $\mathfrak{g} = \mu(\mathbf{z}) \cdot \bar{\mathfrak{g}} \cdot \mu(\mathbf{z})^{-1}$ is the *hidden symmetry algebra* responsible for the QES character of $H(\mathbf{x})$. In this paper we shall be almost exclusively concerned with this special type of quasi-exactly solvable Schrödinger operators, that we shall call *algebraic* to single them out from the rest. The function $\mu(\mathbf{z})$ is usually called the *gauge factor* in the literature, and $\overline{H}(\mathbf{z})$ is referred to as the *gauge Hamiltonian.*

In one dimension, the only Lie algebra of first-order differential operators is (up to equivalence) the standard projective realization of $\mathfrak{sl}(2)$ (or its subalgebras), with basis elements

$$J_N^- = \partial_z, \qquad J_N^0 = z\,\partial_z - \frac{N}{2}, \qquad J_N^+ = z^2\,\partial_z - N\,z\,. \tag{4}$$

If N is a non-negative integer, the latter algebra admits a $(N+1)$-dimensional representation in the space $\mathcal{P}_N$ of polynomials of degree $\leq N$. The most general second-order differential operator belonging to the enveloping algebra of the Lie algebra (4), obtained by constructing an arbitrary quadratic combination of the generators (4), is of the form

$$-\overline{H}(z) = P(z)\,\partial_z^2 + \tilde{Q}(z)\,\partial_z + \tilde{R}(z) \tag{5}$$

with

$$\tilde{Q}(z) = Q(z) - \frac{N-1}{2}\,P'(z)\,, \qquad \tilde{R}(z) = R - \frac{N}{2}\,Q'(z) + \frac{N}{12}(N-1)P''(z)\,, \tag{6}$$

where P, Q and R are arbitrary polynomials of degree 4, 2 and 0, respectively, and the minus sign is for later convenience. If P is positive, the operator (5) is equivalent (in the sense of (2)) to a Schrödinger operator $H(x) = -\partial_x^2 + V(x)$ by the change of variables

$$x = \xi(z) = \int^z \frac{dy}{\sqrt{P(y)}} \tag{7}$$

and conjugation by the gauge factor

$$\mu(z) = P(z)^{-1/4} \exp\left\{ \int^z \frac{\tilde{Q}(y)}{2P(y)}\,dy \right\}. \tag{8}$$

Let $N \geq 1$, and assume, for the sake of simplicity, that the gauge Hamiltonian (5) is diagonalizable in $\mathcal{P}_N$. Then $\overline{H}(z)$ has $N+1$ algebraically computable *polynomial* eigenfunctions $\varphi_k(z)$ $(0 \leq k \leq N)$ of degree $\leq N$, and therefore $H(x)$ has $N+1$ algebraically computable eigenfunctions of the form

$$\psi_k(x) = \mu(z)\,\varphi_k(z)|_{z=\xi^{-1}(x)}\,, \qquad 0 \leq k \leq N\,. \tag{9}$$

Following Calogero's original idea, consider now the time-dependent Schrödinger equation with Hamiltonian $H(x)$, namely

$$H(x)\Psi(x,t) = \mathrm{i}\,\partial_t\Psi(x,t)\,. \tag{10}$$

Since $H(x)$ is time-independent, the latter equation will admit solutions of the form

$$\Psi(x,t) = \sum_{k=0}^{N} c_k\,\psi_k(x)\,\mathrm{e}^{-\mathrm{i}E_k t}\,, \tag{11}$$

where $\psi_k(x)$ is an eigenfunction of $H(x)$ of the form (9) with energy E_k, and $c_0, \dots, c_N$ are arbitrary complex constants. Equivalently,

$$\Psi(x,t) = \mu(z)\Phi(z,t)\big|_{z=\xi^{-1}(x)}\,, \tag{12}$$

with

$$\Phi(z,t) = \sum_{k=0}^{N} c_k\, \varphi_k(z)\, \mathrm{e}^{-\mathrm{i}E_k t}\,. \tag{13}$$

Since each $\varphi_k(z)$ is a polynomial of degree $\leq N$ (and at least one of them is of degree N, since otherwise $\overline{H}(z)$ would not be diagonalizable in $\mathcal{P}_N$), it follows from the latter equation that $\Phi(z,t)$ is a polynomial of degree N in z with time-dependent coefficients. Therefore we can write

$$\Phi(z,t) = C(t)\prod_{j=1}^{N}[z - z_j(t)]\,. \tag{14}$$

It follows from Eq. (12) that each zero $z_j(t)$ of $\Phi(z,t)$ yields a zero

$$x_j(t) = \xi\big(z_j(t)\big)\,, \qquad 1 \leq j \leq N\,, \tag{15}$$

of the time-dependent wavefunction $\Psi(x,t)$. We will study the motion of these zeros, showing that it can be derived from a classical Lagrangian.

Indeed, substituting (12) into Schrödinger's equation (10) and using (14) we easily obtain the equation

$$\sum_{\substack{j,k=1\\ j\neq k}}^{N} \frac{P(z)}{\big(z - z_j(t)\big)\big(z - z_k(t)\big)} + \sum_{j=1}^{N}\frac{\tilde{Q}(z)}{z - z_j(t)} + \tilde{R}(z) = \mathrm{i}\sum_{j=1}^{N}\frac{\dot{z}_j(t)}{z - z_j(t)} - \mathrm{i}\,\frac{\dot{C}(t)}{C(t)}\,, \tag{16}$$

which must hold identically in z and t. Equating the residue of both sides at $z = z_k(t)$ we arrive at the following system of differential equations for the functions $z_k(t)$:

$$\mathrm{i}\,\dot{z}_k = \tilde{Q}(z_k) + 2\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{P(z_k)}{z_k - z_j} \equiv F_k(\mathbf{z})\,, \quad 1 \leq k \leq N\,. \tag{17}$$

Conversely, it can be shown without difficulty that the latter equations imply Eq. (16). In order to show that $(x_1(t), \dots, x_N(t))$ follows a trajectory of a certain Lagrangian system, we shall make use of the following Lemma:

Lemma 1. *Consider the autonomous system of first-order ordinary differential equations*

$$\mathrm{i}\,\dot{x}_k = f_k(\mathbf{x}); \qquad k = 1, \dots, N\,, \quad \mathbf{x} \equiv (x_1, \dots, x_N) \in \mathbf{C}^{\mathbf{N}}\,. \tag{18}$$

If the one-form $\sum_{k=1}^{N} f_k(\mathbf{x})\, dx_k$ is closed, then the second-order system obtained by differentiating (18) *once with respect to t is Lagrangian, with Lagrangian given by*

$$L = \dot{\mathbf{x}}^2 - \sum_{j=1}^{N} f_j^2(\mathbf{x})\,. \tag{19}$$

Proof. The second-order system obtained by differentiating (18) with respect to t is

$$\ddot{x}_k = -\mathrm{i} \sum_{k=1}^{N} \dot{x}_j \frac{\partial f_k}{\partial x_j} = -\sum_{k=1}^{N} f_j \frac{\partial f_k}{\partial x_j}\,, \qquad 1 \le k \le N\,. \tag{20}$$

On the other hand, the Euler–Lagrange equations associated to the Lagrangian (19) are

$$2\,\ddot{x}_k = \frac{\partial L}{\partial x_k} = -2 \sum_{j=1}^{N} f_j \frac{\partial f_j}{\partial x_k}\,, \qquad 1 \le k \le N\,. \tag{21}$$

If $\sum_{k=1}^{N} f_k(\mathbf{x})\, dx_k$ is closed then

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial f_j}{\partial x_k}\,,$$

so that (20) and (21) are indeed identical. *Q.E.D.*

In our case, the system (18) is obtained from (17) by the change of independent variables (15), with $\xi(z)$ as in (7). Since $\dot{x}_k = P(z_k)^{-1/2}\, \dot{z}_k$, we have

$$f_k = P(z_k)^{-1/2}\, F_k \tag{22}$$

and

$$\sum_{k=1}^{N} f_k(\mathbf{x})\, dx_k = \sum_{k=1}^{N} \frac{F_k(\mathbf{z})}{P(z_k)}\, dz_k\,. \tag{23}$$

The latter one-form is clearly closed, since if $j \neq k$ we have

$$\frac{\partial}{\partial z_j}\left(\frac{F_k(\mathbf{z})}{P(z_k)}\right) = \frac{\partial}{\partial z_j}\left[\frac{\tilde{Q}(z_k)}{P(z_k)} + \sum_{\substack{l=1\\ l\neq k}}^{N} \frac{2}{z_k - z_l}\right] = \frac{2}{(z_k - z_j)^2}\,,$$

which is clearly symmetric under the exchange of j and k. It follows from the previous Lemma and Eqs. (19) and (22) that the zeros $\mathbf{x}(t)$ of the time-dependent wavefunction (11) move along a trajectory of the Lagrangian system with Lagrangian

$$L = \sum_{k=1}^{N} \dot{x}_k^2 - \sum_{k=1}^{N} \frac{F_k^2(\mathbf{z})}{P(z_k)}\,, \tag{24}$$

whose associated Hamiltonian is

$$H_N = \sum_{k=1}^N p_k^2 + \sum_{k=1}^N \frac{F_k^2(\mathbf{z})}{P(z_k)} \,. \tag{25}$$

Substituting (17) into (25) and letting $p_k = -\mathrm{i}\,\partial_{x_k}$ we obtain the N-particle quantum Hamiltonian

$$H_N = -\sum_{k=1}^N \partial_{x_k}^2 + V_N(\mathbf{x}) \,, \tag{26}$$

where

$$V_N(\mathbf{x}) = \sum_{k=1}^N \frac{\tilde{Q}_k^2}{P_k} + 4\sum_{\substack{j,k=1\\ j\neq k}}^N \frac{\tilde{Q}_k}{z_{kj}} + 4\sum_{\substack{j,k=1\\ j\neq k}}^N \frac{P_k}{z_{kj}^2} + 4\sum_{\substack{j,k,l=1\\ j\neq k\neq l\neq j}}^N \frac{P_k}{z_{kj}z_{kl}} \tag{27}$$

and we have set

$$P_k = P(z_k)\,, \qquad \tilde{Q}_k = \tilde{Q}(z_k)\,, \qquad z_{kj} = z_k - z_j \,.$$

It will be convenient for our purposes to allow the coefficients of each of the four sums in Eq. (27) to be different. Dropping an inessential overall factor we finally arrive at the following formula for the potential:

$$V_N(\mathbf{x}) = \frac{1}{4}\sum_{k=1}^N \frac{\tilde{Q}_k^2}{P_k} + g_1\sum_{\substack{j,k=1\\ j\neq k}}^N \frac{\tilde{Q}_k}{z_{kj}} + g_2\sum_{\substack{j,k=1\\ j\neq k}}^N \frac{P_k}{z_{kj}^2} + g_3\sum_{\substack{j,k,l=1\\ j\neq k\neq l\neq j}}^N \frac{P_k}{z_{kj}z_{kl}} \,. \tag{28}$$

The main purpose of this paper is to show that, for certain choices of the polynomials $P(z)$ and $\tilde{Q}(z)$ (and, sometimes, the constant g_1), the N-particle quantum Hamiltonian with potential (28) is quasi-exactly solvable or even, in certain cases, exactly solvable. This will be proved in the next Section, essentially by showing that H_N lies in the enveloping algebra of a quasi-exactly solvable Lie algebra of first-order differential operators equivalent to a certain standard realization of $\mathfrak{sl}(N+1)$ (cf. Eq. (29) below).

3 Algebraization of H_N

We shall prove in this Section that the Hamiltonian (26)–(28) derived in the previous Section is algebraic, provided that $P(z)$, $\tilde{Q}(z)$ and (in certain cases) g_1 are chosen appropriately. In analogy with the one-dimensional case, we shall show that $H_N(\mathbf{x})$ is equivalent under a change of variables and a gauge transformation (2) to a gauge Hamiltonian $\overline{H}_N(\mathbf{z})$ which can be written as a quadratic polynomial in the differential operators

$$\begin{aligned} &\mathcal{D}_k = \partial_{\tau_k}\,, \qquad \mathcal{N}_{jk} = \tau_j\,\partial_{\tau_k}\,, \qquad \mathcal{U}_k = \tau_k\left(m - \sum_{i=1}^N \tau_i\,\partial_{\tau_i}\right); \\ &j,k = 1,2,\dots,N\,, \end{aligned} \tag{29}$$

spanning the Lie algebra $\mathfrak{sl}(N+1)$. In the previous formulas m is a non-negative integer, and we have denoted by

$$\tau_k = \sum_{i_1<i_2<\cdots<i_k} z_{i_1} z_{i_2} \cdots z_{i_k}\,, \qquad 1 \le k \le N\,, \tag{30}$$

the k-th elementary symmetric function. Since the operators (29) preserve the finite-dimensional module

$$\mathcal{M}_m = \operatorname{span}\left\{ \tau_1^{l_1} \tau_2^{l_2} \cdots \tau_N^{l_N} : \sum_{i=1}^N l_i \le m \right\} \tag{31}$$

of polynomials of degree $\le m$ in the symmetric variables $\tau = (\tau_1, \dots, \tau_N)$, $\overline{H}_N(\mathbf{z})$ is algebraic, possessing (at most) $\dim \mathcal{M}_m = \binom{m+N}{m}$ eigenfunctions $\chi_k\big(\tau(\mathbf{z})\big)$ that are polynomials of degree $\le m$ in τ. It follows from the discussion in the previous Section that $H_N(\mathbf{x})$ is also algebraic, and that it admits eigenfunctions of the form

$$\psi_k(\mathbf{x}) = \mu(\mathbf{z})\, \chi_k\big(\tau(\mathbf{z})\big)\,, \tag{32}$$

where $\mu(\mathbf{z})$ is the gauge factor, whose specific form will be given below (cf. (37)), and $\chi_k\big(\tau(\mathbf{z})\big)$ is a polynomial eigenfunction of $\overline{H}_N(\mathbf{z})$ which can be algebraically computed. For this reason, we shall henceforth use the term *algebraic eigenfunction* to refer to an eigenfunction of H_N of the form (32).

The generators

$$\mathcal{D}_k\,, \quad \mathcal{N}_{jk}\,; \qquad 1 \le j, k \le N \tag{33}$$

span the *Borel subalgebra* $\mathfrak{b}_{N+1} \subset \mathfrak{sl}(N+1)$, which preserves the infinite sequence of subspaces $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$. Therefore, if $\overline{H}_N$ is a quadratic combination of the differential operators (33) then the physical Hamiltonian H_N is *exactly solvable.*

It will also prove convenient in what follows to define the polynomial subspaces

$$\mathcal{L}_k = \operatorname{span}\left\{ \tau_1^{l_1} \tau_2^{l_2} \dots \tau_N^{l_N} : \sum_{i=1}^N i\, l_i = k \right\} \tag{34}$$

of all the symmetric polynomials homogeneous of degree k in the variables $\mathbf{z} = (z_1, \dots, z_N)$, and their direct sums

$$\hat{\mathcal{M}}_m = \bigoplus_{k=0}^m \mathcal{L}_k\,. \tag{35}$$

Clearly, $\hat{\mathcal{M}}_m \subset \mathcal{M}_m$, though in general $\hat{\mathcal{M}}_m$ need not be invariant under the action of H_N. An important exception occurs when H_N is exactly solvable; indeed, it will be shown in Section 4 that in this case H_N preserves the infinite sequence $\hat{\mathcal{M}}_0 \subset \hat{\mathcal{M}}_1 \subset \hat{\mathcal{M}}_2 \subset \dots$. This important fact shall be used in Section 4 to derive a formula for the energy spectrum of all the exactly solvable models we shall construct, and in Section 5 to exactly compute some eigenfunctions of H_N and their corresponding energies for an arbitrary number of particles N.

We shall restrict ourselves in this paper to polynomials $P(z)$ whose corresponding one-dimensional QES operators (5) admit *normalizable* (i.e, square-integrable) eigenfunctions, [12], leaving the periodic cases for a future work. In this case, there are 5 canonical forms for $P(z)$, all of which can be taken as monic polynomials of degree not greater than 2. By (6), $\tilde{Q}(z)$ is then an arbitrary quadratic polynomial

$$\tilde{Q}(z) = \tilde{c}_+ \, z^2 + \tilde{c}_0 \, z + \tilde{c}_- \, . \tag{36}$$

Let us define the gauge Hamiltonian $\overline{H}_N(\mathbf{z})$ by Eq. (2), where the gauge factor

$$\mu(\mathbf{z}) = \prod_{j<k} z_{jk}^a \cdot \prod_k P_k^{\frac{b}{2}} \, \exp\left\{ \frac{c}{2} \int^{z_k} \frac{\tilde{Q}}{P} \right\} \tag{37}$$

has been chosen by analogy with the one-dimensional case (cf. [12]), and a, b, c are real parameters. (From now on, all indices in sums and products will implicitly run from 1 to N, with the restrictions indicated under the summation or product symbols). Using (28) and dropping an additive constant we obtain the following explicit formula for $\overline{H}_N(\mathbf{z})$:

$$\overline{H}_N(\mathbf{z}) = -\sum_k P_k \, \partial_{z_k}^2 - \sum_k \left[\left(b + \frac{1}{2} \right) P_k' + c \, \tilde{Q}_k \right] \partial_{z_k} - 2a \sum_{j \neq k} \frac{P_k}{z_{kj}} \, \partial_{z_k} + \overline{V}_N(\mathbf{z}) \, , \tag{38}$$

where

$$\overline{V}_N(\mathbf{z}) = A_1 \, \tau_1 + A_2 \sum_{j \neq k} \frac{P_k}{z_{kj}^2} + \sum_k \frac{1}{P_k} \left[A_3 \, P_k'^{\,2} + A_4 \, \tilde{Q}_k \, P_k' + A_5 \, \tilde{Q}_k^2 \right] , \tag{39}$$

and the coefficients A_i are given by

$$\begin{aligned} &A_1 = \tilde{c}_+ \left[(N-1)(g_1 - ac) - c \right] , \qquad A_2 = g_2 - a(a-1) \, , \qquad A_3 = -\frac{1}{4} \, b \, (b-1) \, , \\ &A_4 = -\frac{c}{4} \, (2b - 1) \, , \qquad\qquad\qquad\quad A_5 = \frac{1}{4} \, (1 - c^2) \, . \end{aligned} \tag{40}$$

We want to find sufficient conditions for $\overline{H}_N$ to be expressible as a quadratic combination of the generators (29). From (39) it is clear that one such condition is $A_2 = 0$, which yields the following relation between the coupling constant g_2 and the exponent a in the gauge factor (37):

$$g_2 = a(a-1) \, . \tag{41}$$

The differential part of $\overline{H}_N$ equals

$$\begin{aligned} -m \, c \, \tilde{c}_+ \, \tau_1 + c \, \tilde{c}_+ \, \mathcal{U}_1 \quad &\mod \mathfrak{b}_{N+1} \\ = -m \, c \, \tilde{c}_+ \, \tau_1 \quad &\mod \mathfrak{sl}(N+1) \, . \end{aligned} \tag{42}$$

In Cases 1–3, $P(z) = z^2 + \epsilon$ with $\epsilon = 0, \pm 1$, and the last term in (39) can be expressed (up to a constant) as

$$A_5\,\tilde{c}_+^2 \sum_k z_k^2 + 2\tilde{c}_+(A_4 + \tilde{c}_0 A_5)\,\tau_1 + \sum_k \frac{\rho_1 z_k + \rho_0}{P_k}\,. \tag{43}$$

for certain constants ρ_0 and ρ_1. It is then clear that the terms proportional to τ_1 in (39) and (42) must cancel, and the remaining terms in (43) must vanish, giving rise to the conditions

$$A_5\,\tilde{c}_+ = 0 \tag{44}$$

$$\rho_0 = \rho_1 = 0 \tag{45}$$

$$\tilde{c}_+\left[(N-1)(g_1 - ac) - \left(b + m + \frac{1}{2}\right)c\right] = 0\,. \tag{46}$$

The analogous conditions for Cases 4 and 5 (when $P(z) = z$ and $P(z) = 1$, respectively) can be found at the end of this Section, when we discuss those cases in detail.

In particular, if $\tilde{c}_+ = 0$ then the "quantization condition" (46) is automatically satisfied and the term proportional to the generator $\mathcal{U}_1$ in (42) vanishes (this also holds in Cases 4 and 5), so that in this case $\overline{H}_N$ is a quadratic combination of operators belonging to the Borel subalgebra of $\mathfrak{sl}(N+1)$. Therefore the condition $\tilde{c}_+ = 0$ ensures the *exact solvability* of the potential (28). The specific conditions for $\overline{H}_N$ to be algebraic, together with the explicit form of the potential (28) and the normalizability conditions for the algebraic eigenfunctions, will be discussed on a case by case basis below.

All the potentials discussed in what follows diverge when $x_j = x_k$ (cf. Eq. (28)). For this reason, we shall consider them to be defined in the open set

$$x_N < x_{N-1} < \cdots < x_1\,. \tag{47}$$

Moreover, the finiteness of the mean kinetic energy of the algebraic eigenfunctions near the hyperplanes $x_j = x_k$ requires that $a > 1/2$, so that $g_2 > -\frac{1}{4}$ in all cases. In Cases 2 and 4 we shall impose the additional restriction $x_k > 0$ $(1 \le k \le N)$ to make the change of variables (7) one-to-one ((28) is generically singular at $x_k = 0$ in these cases), so that the potential is defined in the open subset

$$0 < x_N < x_{N-1} < \cdots < x_1\,. \tag{48}$$

This corresponds to choosing a fundamental chamber for the action of the Weyl group, [5].

The following constants will be used throughout this Section:

$$C_\pm = c\,\tilde{c}_\pm\,, \qquad C_0 = c\,\tilde{c}_0\,, \tag{49}$$

$$B = b + \frac{1}{2}c\,\tilde{c}_0\,, \qquad B_- = b + c\,\tilde{c}_-\,. \tag{50}$$

3.1 Case 1. $P_k = z_k^2 + 1\,,\ \ z_k = \sinh x_k$.

It can be shown that a necessary and sufficient condition for all the wavefunctions in the algebraic sector to be normalizable is

$$\tilde{c}_+ = 0\,, \tag{51}$$
$$a(N-1) + B + m < 0\,; \tag{52}$$

hence (44) and (46) are automatically satisfied, and the potential in this case is *exactly solvable.* Conditions (45) reduce to

$$\left(B - \frac{1}{2}\right)^2 = \frac{1}{4}\left(1 + C_-^2\right) - \eta_0\,, \tag{53}$$
$$C_-\left(B - \frac{1}{2}\right) = \eta_1\,, \tag{54}$$

where η_0 and η_1 are defined by

$$\eta_0 = \frac{1}{4}\left(\tilde{c}_-^2 - \tilde{c}_0^2\right), \qquad \eta_1 = \frac{1}{2}\,\tilde{c}_0\,\tilde{c}_-\,. \tag{55}$$

The solution to the previous system becomes unique after imposing the normalizability condition (52), and is given by

$$B = \frac{1}{2}\left(1 - \sqrt{s}\right), \qquad C_- = -\frac{2\eta_1}{\sqrt{s}}\,, \tag{56}$$

with

$$\frac{s}{2} = \frac{1}{4} - \eta_0 + \sqrt{\left(\frac{1}{4} - \eta_0\right)^2 + \eta_1^2}\,. \tag{57}$$

The potential (28) is in this case

$$V_N(\mathbf{x}) = \sum_k \left(\eta_0 + \eta_1\,\sinh x_k\right)\operatorname{sech}^2 x_k + g_2\sum_{j\neq k}\frac{\cosh^2 x_k}{(\sinh x_j - \sinh x_k)^2}\,, \tag{58}$$

which, after dropping the constant term $g_2N(N-1)/2$, can be expressed in the more familiar form

$$V_N(\mathbf{x}) = \sum_k \left(\eta_0 + \eta_1\,\sinh x_k\right)\operatorname{sech}^2 x_k + \frac{g_2}{4}\sum_{j\neq k}\left[\operatorname{csch}^2\left(\frac{x_j - x_k}{2}\right) - \operatorname{sech}^2\left(\frac{x_j + x_k}{2}\right)\right]. \tag{59}$$

In the latter formula, η_0 and η_1 can be taken as arbitrary real parameters (cf. (55)) restricted only by the normalizability condition (52), which can be written as

$$m < \frac{1}{2}\left(\sqrt{s}-1\right) - a(N-1)\,, \tag{60}$$

with s defined by Eq. (57). The exactly solvable potential (59) will admit square-integrable algebraic eigenfunctions if the latter inequality can be satisfied by a non-negative integer m, i.e, if the right-hand side is positive, which yields the restriction

$$s > [1+2a(N-1)]^2\,. \tag{61}$$

The algebraic eigenfunctions (32) in this case have the form

$$\psi(\mathbf{x}) = \chi(\tau)\cdot\prod_{j<k}\left(\cosh\frac{x_j+x_k}{2}\right)^a\left(\sinh\frac{x_j-x_k}{2}\right)^a\cdot\prod_k \operatorname{sech}^{|B|}x_k\,\mathrm{e}^{\frac{C_-}{2}\arctan(\sinh x_k)}\,, \tag{62}$$

where a, B, and C_- are given by Eqs. (41), (56), and (57). The function $\chi(\tau)$ is a polynomial eigenfunction of $\overline{H}_N$ of degree $\leq m$ in the symmetric variables τ_k, where m is the largest non-negative integer satisfying (60).

The potential (59) is not, as it stands, one of Olshanetsky–Perelomov's completely integrable models. If we perform the imaginary translation $x_k \mapsto x_k + \mathrm{i}\,\frac{\pi}{2}$ $(1\leq k\leq N)$, then (59) becomes

$$\begin{aligned}V_N(\mathbf{x}) = \frac{1}{4}\sum_k&\left[\frac{\beta_+(\beta_+-1)}{\sinh^2\frac{x_k}{2}} - \frac{\beta_-(\beta_--1)}{\cosh^2\frac{x_k}{2}}\right]\\ &+\frac{g_2}{4}\sum_{j\neq k}\left[\operatorname{csch}^2\left(\frac{x_j+x_k}{2}\right)+\operatorname{csch}^2\left(\frac{x_j-x_k}{2}\right)\right],\end{aligned} \tag{63}$$

with

$$\beta_\pm = B \mp \mathrm{i}\,\frac{C_-}{2}\,, \tag{64}$$

which is a completely integrable potential of BC_N type. Note, however, that the coefficients $\beta_\pm(\beta_\pm-1)$ are, generally speaking, *complex* for real values of B and C_-. In fact, these coefficients can be real only in two cases: i) $C_-=0$, and ii) $B=1/2$. In the first case, both parameters $\beta_\pm = B$ are real and equal, and (63) reduces to the well-known completely integrable potential of C_N type

$$V_N(\mathbf{x}) = B(B-1)\sum_k \operatorname{csch}^2 x_k + \frac{g_2}{4}\sum_{j\neq k}\left[\operatorname{csch}^2\left(\frac{x_j+x_k}{2}\right)+\operatorname{csch}^2\left(\frac{x_j-x_k}{2}\right)\right]. \tag{65}$$

Indeed, in this case the algebraic eigenfunctions (62) reduce to

$$\psi(\mathbf{x}) = \chi(\tau)\cdot\prod_{j<k}\left(\sinh\frac{x_j+x_k}{2}\,\sinh\frac{x_j-x_k}{2}\right)^a\cdot\prod_k\sinh^B x_k\,, \tag{66}$$

thus recovering the results in Ref. [5]. The second case is not interesting, since $B = 1/2$ implies

$$\beta_\pm(\beta_\pm - 1) = -\frac{1}{4}\,(1 + C_-^2)\,. \tag{67}$$

Therefore in this case the Hamiltonian is not self-adjoint, since the potential behaves as

$$-\frac{1}{4}\,(1 + C_-^2)\,x_k^{-2} < -\frac{1}{4}\,x_k^{-2} \tag{68}$$

as $x_k \to 0$. Thus when $C_- \neq 0$, i.e, when $\eta_1 \neq 0$, the exactly solvable potential (59) does not seem to be directly related to a root system.

3.2 Case 2. $P_k = z_k^2 - 1\,, \quad z_k = \cosh x_k$.

In this case, the normalizability of the algebraic eigenfunctions requires that either

$$C_+ < 0 \tag{69}$$

or

$$C_+ = 0 \quad \text{and} \quad a(N-1) + B + m < 0\,, \tag{70}$$

which leads to two different subcases, the first one consisting of QES and the second one of ES potentials. We shall therefore discuss separately the two cases $C_+ < 0$ and $C_+ = 0$.

3.2.1 Case 2.a. $C_+ < 0$.

Conditions (44) and (45) reduce in this case to

$$c^2 = 1 \tag{71}$$

$$C_+ + C_- = 0 \tag{72}$$

$$B(B-1) = \frac{C_0^2}{4}, \tag{73}$$

while condition (46) can be expressed as

$$c(N-1)g_1 = a(N-1) + m + b + \frac{1}{2}\,. \tag{74}$$

Imposing regularity of the wavefunctions on the hyperplanes $x_k = 0$ forces a unique solution to Eq. (73), given by

$$B = \frac{1}{2}\left(1 + \sqrt{1 + C_0^2}\right). \tag{75}$$

Taking into account Eqs. (71)–(74), the potential (28) can be written as

$$V_N(\mathbf{x}) = \sum_k \left[\frac{C_+^2}{4} \sinh^2 x_k + C_+ \left(a(N-1) + B + m + \frac{1}{2} \right) \cosh x_k + \frac{C_0^2}{4} \operatorname{csch}^2 x_k \right] + \frac{g_2}{4} \sum_{j \neq k} \left[\operatorname{csch}^2 \left(\frac{x_j + x_k}{2} \right) + \operatorname{csch}^2 \left(\frac{x_j - x_k}{2} \right) \right]. \quad (76)$$

This potential is *quasi-exactly solvable,* and it has $\dim \mathcal{M}_m = \binom{m+N}{m}$ algebraic eigenfunctions of the form

$$\psi(\mathbf{x}) = \chi(\tau) \cdot \prod_{j<k} \left(\sinh \frac{x_j + x_k}{2} \sinh \frac{x_j - x_k}{2} \right)^a \cdot \prod_k (\sinh x_k)^B \, \mathrm{e}^{\frac{C_+}{2} \cosh x_k} , \quad (77)$$

where B is given by Eq. (75), and $\chi(\tau)$ is a polynomial eigenfunction of $\overline{H}_N$ of degree $\leq m$ in the symmetric variables τ_k. The potential (76) is regular on the hyperplanes $x_k = 0$ if and only if $C_0 = 0$. In this case $B = 1$, and therefore the algebraic eigenfunctions (77) are naturally extended as odd functions to the region

$$|x_N| < |x_{N-1}| < \cdots < |x_1| . \quad (78)$$

A potential somewhat more general than (76) has been proved to be completely integrable in the *classical* case by Inozemtsev and Meshcheryakov, [13], [14]. Its quantum integrability has been conjectured by the same authors in Ref. [11], although, to the best of our knowledge, no proof of this fact has been given. We have shown in this Section that the potential (76), which belongs to the general class considered in Ref. [11], is *quasi-exactly solvable.* In fact, we shall explicitly compute in Section 5 some eigenfunctions and eigenvalues of the Hamiltonian with potential (76) for $m = 1$ and $N = 3$ particles.

3.2.2 Case 2.b. $C_+ = 0$.

In this case the potential (28), which is *exactly solvable*, can be written as

$$V_N(\mathbf{x}) = \frac{1}{4} \sum_k \left[\eta_+ \operatorname{csch}^2 \Big(\frac{x_k}{2} \Big) - \eta_- \operatorname{sech}^2 \Big(\frac{x_k}{2} \Big) \right] + \frac{g_2}{4} \sum_{j \neq k} \left[\operatorname{csch}^2 \left(\frac{x_j + x_k}{2} \right) + \operatorname{csch}^2 \left(\frac{x_j - x_k}{2} \right) \right], \quad (79)$$

where

$$\eta_\pm = \frac{1}{4} \left(\tilde{c}_- \pm \tilde{c}_0 \right)^2 \geq 0 . \quad (80)$$

Conditions (44) and (46) are now automatically satisfied, while (45) reduces to

$$\left(B - \frac{1}{2} \right)^2 = \frac{1}{4} \left(1 - C_-^2 \right) + \frac{1}{2} \left(\eta_+ + \eta_- \right) \quad (81)$$

$$C_- \left(B - \frac{1}{2} \right) = \frac{1}{2} \left(\eta_+ - \eta_- \right) . \quad (82)$$

Introducing the parameters

$$\beta_\pm = B \pm \frac{C_-}{2}\,, \tag{83}$$

the latter system reduces to

$$\beta_\pm(\beta_\pm - 1) = \eta_\pm\,, \tag{84}$$

which uniquely determines $\beta_\pm$. Indeed, the algebraic eigenfunctions in this case are of the form

$$\psi(\mathbf{x}) = \chi(\tau)\cdot\prod_{j<k}\Big(\sinh\frac{x_j+x_k}{2}\,\sinh\frac{x_j-x_k}{2}\Big)^a\cdot\prod_k \sinh^{\beta_+}\Big(\frac{x_k}{2}\Big)\,\cosh^{\beta_-}\Big(\frac{x_k}{2}\Big)\,, \tag{85}$$

where $\chi(\tau)$ is a polynomial eigenfunction of $\overline{H}_N$ of degree $\leq m$ in the symmetric variables τ. Therefore $\beta_+ \geq 1/2$ for the eigenfunctions to be regular (with finite kinetic energy) on the hyperplanes $x_k = 0$, while $\beta_- \leq -1/2$ is necessary for the normalizability condition (70), which now reads

$$\frac{1}{2}\,(\beta_+ + \beta_-) + a(N-1) + m < 0\,, \tag{86}$$

to hold. Thus Eqs. (84) have the unique solution

$$\beta_\pm = \frac{1}{2}\Big(1 \pm \sqrt{1+4\,\eta_\pm}\Big)\,, \tag{87}$$

from which it actually follows that $\beta_+ \geq 1$ and $\beta_- \leq -1$.

The exactly solvable potential (79) is known to be integrable, [5], and is associated with the root system BC_N, when $\eta_+ \neq \eta_-$, or C_N, when $\eta_+ = \eta_-$. The potential is regular on the hyperplanes $x_k = 0$ if and only if $\beta_+ = 1$, in which case the algebraic eigenfunctions (85) are naturally extended as odd functions to the region (78).

3.3 Case 3. $P_k = z_k^2\,,\ \ z_k = \mathrm{e}^{x_k}$.

For the algebraic eigenfunctions to be square-integrable, one of the following three conditions must be satisfied:

1. $C_+ < 0\,,\ C_- > 0$
2. $C_+ = 0\,,\ C_- > 0\,,\ a(N-1) + B + m < 0$
3. $C_+ < 0\,,\ C_- = 0\,,\ B > 0$

In principle, the first case yields a family of QES potentials and the remaining two cases a family of ES ones. However, it can be shown that the potential in the third case can be obtained from that in the second one by the reflection $\mathbf{x} \mapsto -\mathbf{x}$. We shall therefore consider only the first two cases.

3.3.1 Case 3a. $C_+ < 0\,,\ \ C_- > 0.$

Conditions (44) and (45) are satisfied if

$$c^2 = 1\,, \quad b = \frac{1}{2}\,, \tag{88}$$

and the quantization condition (46) can be written as

$$c\,g_1\,(N-1) = a(N-1) + b + m + \frac{1}{2}\,. \tag{89}$$

The potential (28) becomes in this case

$$\begin{aligned} V_N(\mathbf{x}) = \sum_k \bigg[\frac{C_-^2}{4}\,\mathrm{e}^{-2x_k} + \frac{1}{2} C_0\,C_-\mathrm{e}^{-x_k} + C_+\big(a(N-1) + \frac{C_0}{2} + m + 1\big)\mathrm{e}^{x_k} + \frac{C_+^2}{4}\,\mathrm{e}^{2x_k} \bigg] \\ + \frac{g_2}{4} \sum_{j\neq k} \operatorname{csch}^2\left(\frac{x_j - x_k}{2}\right). \end{aligned} \tag{90}$$

The quantum Hamiltonian (26) with potential (90) defined in the region (47) is *quasi-exactly solvable.* It possesses $\binom{m+N}{m}$ algebraic eigenfunctions of the form

$$\psi(\mathbf{x}) = \chi(\tau) \cdot \prod_{j<k} (\mathrm{e}^{x_j} - \mathrm{e}^{x_k})^a \cdot \prod_k \mathrm{e}^{\frac{1}{2}(C_0+1)x_k}\,\mathrm{e}^{\frac{1}{2}\left(C_+\mathrm{e}^{x_k} - C_-\mathrm{e}^{-x_k}\right)}, \tag{91}$$

which can be more conveniently written as

$$\psi(\mathbf{x}) = \chi(\tau) \cdot \prod_{j<k} \sinh^a\left(\frac{x_j - x_k}{2}\right) \cdot \prod_k \mathrm{e}^{\frac{1}{2}[C_0 + a(N-1)+1]x_k}\,\mathrm{e}^{\frac{1}{2}\left(C_+\mathrm{e}^{x_k} - C_-\mathrm{e}^{-x_k}\right)}, \tag{92}$$

where $\chi(\tau)$ is a polynomial eigenfunction of $\overline{H}_N$ of degree $\leq m$ in the symmetric variables τ_k.

The potential (90) with arbitrary constants multiplying each exponential has been shown to be classically completely integrable in Ref. [13], but the explicit solutions of the equations of motion could only be calculated when certain restrictions on the constants were imposed, and only for a subset of initial conditions, [15]. In the quantum case, no exact formulas for the eigenfunctions and their corresponding energies are available when both the negative and positive exponentials are present, although these models have been conjectured to be completely integrable also in this case, [11]. We have proved that for certain values of the coefficients multiplying the exponentials the potential (90) is *quasi-exactly solvable,* and thus some of its energies and eigenfunctions (in general corresponding to excited states) can be exactly computed. See Section 5 for a concrete example when $m = 1$.

3.3.2 Case 3b. $C_+ = 0\,, \quad C_- > 0$.

The conditions (44) and (45) lead again to (88), but now the quantization condition (46) is absent. The potential can be simply obtained by making $C_+ = 0$ in (90):

$$V_N(\mathbf{x}) = \sum_k \left[\frac{C_-^2}{4}\,\mathrm{e}^{-2x_k} + \frac{1}{2} C_0\, C_- \mathrm{e}^{-x_k} \right] + \frac{g_2}{4} \sum_{j \neq k} \operatorname{csch}^2 \left(\frac{x_j - x_k}{2} \right). \tag{93}$$

Note that this potential depends effectively only on one parameter, since the ratio $|C_-/C_0|$ can be assigned any prescribed positive value by performing a suitable translation of the coordinates. The same argument shows that the discrete spectrum of the potential (93) cannot depend on C_-.

The quantum Hamiltonian (26) with potential (93) is *exactly solvable*. It describes an external Morse potential acting on each particle, plus an A_N-type hyperbolic interaction. The discrete spectrum of this potential has been computed by Inozemtsev and Meshcheryakov, [11], by relating (93) to the integrable BC_N potential (79) through a certain formal limit.

The square-integrability of the algebraic eigenfunctions is ensured if Eq. (52) holds or, taking into account (88),

$$C_0 + 2a(N-1) + 2m + 1 < 0\,. \tag{94}$$

This inequality can be satisfied for non-negative integer values of m provided that

$$C_0 + 2a(N-1) + 1 < 0\,. \tag{95}$$

In particular, note that the latter inequality implies that C_0 is negative, and therefore the coefficient of e^{-x_k} in the potential is negative as well. The algebraic eigenfunctions are given in this case by

$$\psi(\mathbf{x}) = \chi(\tau) \cdot \prod_{j<k} \sinh^a \left(\frac{x_j - x_k}{2} \right) \cdot \prod_k \exp \left(\frac{1}{2} [C_0 + a(N-1) + 1] x_k - \frac{C_-}{2} \mathrm{e}^{-x_k} \right), \tag{96}$$

where $\chi(\tau)$ is a polynomial eigenfunction of $\overline{H}_N$ of degree $\leq m$ in τ, m being the largest non-negative integer compatible with (94).

The limit $C_- = 0$ corresponds to one of Olshanetsky–Perelomov's integrable potentials associated to the A_N root system, namely

$$V_N(\mathbf{x}) = \frac{1}{4} a(a-1) \sum_{j \neq k} \operatorname{csch}^2 \left(\frac{x_j - x_k}{2} \right), \tag{97}$$

whose algebraic eigenfunctions are no longer normalizable (indeed, this potential is known to have no bound states, [5]).

3.4 Case 4. $P_k = z_k\,, \quad z_k = \dfrac{x_k^2}{4}$.

The algebraic eigenfunctions will be square integrable provided that

$$C_+ < 0 \tag{98}$$

or

$$C_+ = 0\,, \quad C_0 < 0\,. \tag{99}$$

Conditions (44)–(46) must be modified in this case (since P is no longer quadratic), but after applying similar considerations it can be shown that the operator $\overline{H}_N$ is algebraic provided that

$$c^2 = 1 \tag{100}$$
$$B_-(B_- - 1) = C_-^2 \tag{101}$$
$$C_+\left[(a - c\,g_1)(N-1) + m + \frac{b}{2} + \frac{3}{4}\right] = 0\,. \tag{102}$$

The potential is given by

$$V_N(\mathbf{x}) = \sum_k \left[\eta_0^2\, x_k^6 + \eta_1\, x_k^4 + \eta_2\, x_k^2 + \frac{\eta_3}{x_k^2}\right] + g_2 \sum_{j\neq k}\left[\frac{1}{(x_j+x_k)^2} + \frac{1}{(x_j-x_k)^2}\right]\,, \tag{103}$$

where

$$\begin{aligned} \eta_0 &= -\frac{C_+}{16}\,, \qquad \eta_1 = \frac{C_0 C_+}{32}\,,\\ \eta_2 &= \frac{C_0^2}{16} + \frac{C_+}{4}\left[a(N-1) + m + \frac{B_-}{2} + \frac{3}{4}\right]\,,\\ \eta_3 &= C_-^2 \geq 0\,, \end{aligned} \tag{104}$$

and

$$B_- = \frac{1}{2}\left(1 + \sqrt{1 + 4\,\eta_3}\right)\,. \tag{105}$$

(We have taken the positive square root to ensure that the algebraic eigenfunctions are regular when $x_k \to 0$.) When $C_+ \neq 0$ the potential (103) is *quasi-exactly solvable*. The coefficients in the potential are not all independent in this case, but satisfy the relation

$$\eta_2 = \frac{\eta_1^2}{4\eta_0^2} - 4\eta_0\left[a(N-1) + m + \frac{B_-}{2} + \frac{3}{4}\right]\,. \tag{106}$$

The algebraic eigenfunctions are of the form

$$\psi(\mathbf{x}) = \chi(\tau)\cdot\prod_{j<k}\left(x_j^2 - x_k^2\right)^a \cdot \prod_k x_k^{B_-}\,\mathrm{e}^{-\frac{1}{4\eta_0}\left(\eta_0^2\, x_k^4 + \eta_1\, x_k^2\right)}\,, \tag{107}$$

where B_- is given by (105), and $\chi(\tau)$ is a polynomial eigenfunction of $\overline{H}_N$ of degree $\leq m$ in τ. This QES potential has been recently obtained in somewhat less generality by Hou and Shifman, [10].

If $C_+ = 0$ then the potential (103) becomes *exactly solvable*, and assumes the simpler form

$$V_N(\mathbf{x}) = \sum_k \left(\frac{\omega^2}{4}\, x_k^2 + \frac{\eta_3}{x_k^2} \right) + g_2 \sum_{j \neq k} \left[\frac{1}{(x_j + x_k)^2} + \frac{1}{(x_j - x_k)^2} \right], \tag{108}$$

where now the parameters $\omega \equiv 2\sqrt{\eta_2} = |C_0|/2 > 0$ and $\eta_3 \geq 0$ are independent. The algebraic eigenfunctions are in this case

$$\psi(\mathbf{x}) = \chi(\tau) \cdot \prod_{j<k} \left(x_j^2 - x_k^2 \right)^a \cdot \prod_k x_k^{B_-} \mathrm{e}^{-\frac{\omega}{4} x_k^2}\,, \tag{109}$$

where B_- and $\chi(\tau)$ are as before. This potential is well known to be integrable, [5], and is commonly known as a rational B_N-type potential with harmonic force.

The potentials (103)–(108) are regular on the hyperplanes $x_k = 0$ provided that $\eta_3 = 0$, or, equivalently, $B_- = 1$. By (107)–(109), in this case the algebraic eigenfunctions can again be extended as odd eigenfunctions to the region (78).

Following the idea of Minzoni, Rosenbaum and Turbiner, [16], we can obtain a different QES deformation of the ES potential (108). Note that the gauge hamiltonian $\overline{H}_N$ in this case can be written as

$$\overline{H}_N = h^{(1)} + h^{(2)}\,,$$

where

$$h^{(1)} = -\tau_1\, \partial_{\tau_1}^2 - (A + C_0\, \tau_1)\, \partial_{\tau_1}\,, \tag{110}$$

$$A \equiv N \left[B_- + \frac{1}{2} + a(N-1) \right], \tag{111}$$

depends only on τ_1, and

$$h^{(2)} \cdot \phi(\tau_1) = 0\,.$$

If we consider eigenfunctions χ of $\overline{H}_N$ depending only on τ_1 then the problem reduces to the one-dimensional equation

$$-\tau_1 \chi'' - (A + C_0 \tau_1)\chi' = E\chi\,. \tag{112}$$

If we add a function $U(\tau_1)$ to the potential (108), we can still look for eigenfunctions of $\overline{H}_N$ depending only on τ_1, which must satisfy the equation

$$\tau_1 \chi'' + (A + C_0 \tau_1)\chi' + [E - U(\tau_1)]\chi = 0\,. \tag{113}$$

A wide class of exact solutions of (113) is obtained when the operator $h = h^{(1)} + U(\tau_1)$ is equivalent under a gauge transformation to a quasi-exactly solvable operator on the line. The most general such operator equivalent to h must be of the form

$$\bar{h} = -\tau_1\, \partial_{\tau_1}^2 + (2\gamma\, \tau_1^2 + \beta\, \tau_1 + \alpha)\, \partial_{\tau_1} - 2n\gamma\, \tau_1 + E_0\,, \tag{114}$$

where E_0 is a constant and n is a non-negative integer, [12]. The latter operator leaves invariant the space of polynomials in τ_1 of degree not greater than n, and consequently admits $n+1$ algebraic eigenfunctions. For h and $\bar{h}$ to be equivalent we must have

$$\bar{h} = \mathrm{e}^{-f(\tau_1)} \cdot h \cdot \mathrm{e}^{f(\tau_1)} \,. \tag{115}$$

From this relation we obtain the equations

$$2\tau_1 \, f' + A + C_0 \, \tau_1 = -\left(2\gamma \, \tau_1^2 + \beta \, \tau_1 + \alpha\right) \tag{116}$$

$$\tau_1(f'' + f'^2) + (A + C_0 \, \tau_1) f' - U = 2n\gamma \, \tau_1 - E_0 \,, \tag{117}$$

from which we easily get

$$f = -\frac{1}{2} \left[\gamma \, \tau_1^2 + (\beta + C_0)\tau_1 + (A + \alpha) \log \tau_1\right] \tag{118}$$

and (after dropping a constant term)

$$U(\tau_1) = \gamma^2 \, \tau_1^3 + \beta\gamma \, \tau_1^2 + \frac{1}{4} \left[\beta^2 - C_0^2 + 4\gamma(\alpha - 2n - 1)\right] \tau_1 - \frac{(A - \alpha - 2)(A + \alpha)}{4 \, \tau_1} \,. \tag{119}$$

If we define

$$r^2 = \sum_k x_k^2 = 4 \, \tau_1 \,,$$

then we have shown that the N-particle potential

$$\begin{aligned}
\tilde{V}_N(\mathbf{x}) &= V_N(\mathbf{x}) + U(r^2/4) \\
&= \tilde{A} \, r^6 + \tilde{B} \, r^4 + \tilde{C} \, r^2 + \frac{\tilde{D}}{r^2} + B_-(B_- - 1) \sum_k \frac{1}{x_k^2} \\
&\quad + a(a-1) \sum_{j \neq k} \left[\frac{1}{(x_j + x_k)^2} + \frac{1}{(x_j - x_k)^2}\right] ,
\end{aligned} \tag{120}$$

with

$$\tilde{A} = \frac{\gamma^2}{64} \,, \qquad \tilde{C} = \frac{1}{16} \left[\beta^2 + 4\gamma(\alpha - 2n - 1)\right] , \tag{121}$$

$$\tilde{B} = \frac{\beta\gamma}{16} \,, \qquad \tilde{D} = -(A - \alpha - 2)(A + \alpha), \tag{122}$$

possesses $n+1$ algebraic eigenfunctions of the form

$$\psi_k(\mathbf{x}) = r^{-(A+\alpha)} \, \exp\left(-\frac{\gamma}{32} \, r^4 - \frac{\beta}{8} \, r^2\right) \, \sigma_k(r^2/4) \cdot \prod_{j<k} (x_j^2 - x_k^2)^a \cdot \prod_k x_k^{B_-} \,, \tag{123}$$

where $\sigma_k(\tau_1)$ is a polynomial eigenfunction of $\bar{h}$ of degree less than or equal to n. These eigenfunctions are square-integrable and regular at the origin provided that

$$\gamma > 0 \quad (\text{or } \gamma = 0 \text{ and } \beta > 0), \qquad \alpha > \frac{N}{2} \,, \qquad B_- > \frac{1}{2} \,. \tag{124}$$

3.5 Case 5. $\quad P_k = 1\,,\ \ z_k = x_k$.

The conditions for $\overline{H}_N$ to be algebraic reduce in this case to

$$A_5\,\tilde{c}_+ = A_5\,\tilde{c}_0 = 0\,. \tag{125}$$

On the other hand, the necessary and sufficient conditions for the algebraic eigenfunctions to be normalizable are

$$C_+ = 0\,, \quad C_0 < 0\,. \tag{126}$$

Hence the all potentials in this class are *exactly solvable.* From Eqs. (125) and (126) it follows that $c^2 = 1$, so that the potential (39) reduces to the celebrated Calogero model

$$V_N(\mathbf{x}) = \omega^2 \sum_k x_k^2 + g_2 \sum_{j\neq k} \frac{1}{(x_j - x_k)^2}\,, \tag{127}$$

with

$$\omega = \frac{|C_0|}{2} > 0\,. \tag{128}$$

(To obtain the previous formula, we have performed a constant translation to get rid of the irrelevant parameter C_-.) The algebraic eigenfunctions are of the form

$$\psi(\mathbf{x}) = \chi\big(\tau(\mathbf{x})\big)\cdot\prod_{j<k}(x_j - x_k)^a \cdot \prod_k \mathrm{e}^{-\frac{\omega}{2}\,x_k^2}\,, \tag{129}$$

where the functions $\chi(\tau)$ are polynomial eigenfunctions of $\overline{H}_N$, known in the literature as the *Calogero polynomials.*

4 Energy Spectrum

In this Section we shall obtain an explicit expression for the energy spectrum of all the ES models derived in the previous Section, discussing some of its basic properties. The key idea in this respect is to show that the ES Hamiltonians preserve not only the subspaces $\mathcal{M}_k$, but also the smaller subspaces $\hat{\mathcal{M}}_k$. We shall then define a lexicographic ordering of the basis elements within each subspace $\hat{\mathcal{M}}_k$ ensuring that the matrix of the restriction of $\overline{H}_N$ to $\hat{\mathcal{M}}_k$ is triangular. The spectrum of $\overline{H}_k$, and therefore that of H_N, will thus simply consist of the diagonal elements of these matrices for $k = 0, 1, 2, \dots$.

4.1 Cases 1–3

The gauge Hamiltonian (38) corresponding to the ES Hamiltonians with potential (59), (79) and (93) can be written as

$$\overline{H}_N(\mathbf{z}) = E_0 - \sum_k (z_k^2 + \epsilon)\,\partial_{z_k}^2 - \sum_k \big[(2B+1)z_k + C_-\big]\,\partial_{z_k} - 2a\sum_{j\neq k}\frac{z_k^2+\epsilon}{z_{kj}}\,\partial_{z_k}\,, \tag{130}$$

where $\epsilon = 1, -1, 0$, respectively, and

$$E_0 = -N\left\{\left[B + \frac{a}{2}(N-1)\right]^2 + \frac{a^2}{12}(N^2-1)\right\}. \tag{131}$$

For the potentials (59) and (79) B and C_- are given by Eqs. (56)–(57) and (83), respectively, while $B = \frac{1}{2}(C_0+1)$ for the potential (93) on account of (88). The expression of $\overline{H}_N$ in terms of the symmetric variables is easily found to be

$$\begin{aligned}\overline{H}_N = E_0 &- \sum_{i,j} A^{(2)}_{ij}\partial_{\tau_i}\partial_{\tau_j} - (2B+1)\sum_j j\tau_j\partial_{\tau_j} - a\sum_j j(2N-j-1)\tau_j\partial_{\tau_j}\\ &- C_-\sum_j (N-j+1)\tau_{j-1}\partial_{\tau_j}\\ &+ \epsilon\left(-\sum_{ij} A^{(0)}_{ij}\partial_{\tau_i}\partial_{\tau_j} + a\sum_j (N-j+1)(N-j+2)\tau_{j-2}\partial_{\tau_j}\right),\end{aligned} \tag{132}$$

where the coefficients $A^{(p)}_{ij}$ can be found in the Appendix.

It follows from the latter expression and the structure of the coefficients $A^{(p)}_{ij}$ that $\overline{H}_N$ preserves the subspaces

$$\hat{\mathcal{M}}_k = \bigoplus_{j=0}^{k} \mathcal{L}_j$$

for arbitrary k, where

$$\mathcal{L}_j = \operatorname{span}\left\{\tau_1^{l_1}\tau_2^{l_2}\dots\tau_N^{l_N} : \sum_{i=1}^{N} i\, l_i = j\right\}.$$

In fact, the first line of (132) leaves the subspace $\mathcal{L}_k$ invariant, while the second and third lines take an element of $\mathcal{L}_k$ into $\mathcal{L}_{k-1}$ and $\mathcal{L}_{k-2}$, respectively.

We now introduce an ordering of the basis of $\hat{\mathcal{M}}_k$ consisting of monomials. First of all, we declare the monomials belonging to $\mathcal{L}_j$ to be less than those in $\mathcal{L}_k$ if $j<k$. Within each $\mathcal{L}_j$, the ordering is then defined as follows: if the monomial $\tau_1^{l_1}\dots\tau_N^{l_N}$ is denoted by the multi-index $l \equiv (l_1, l_2, \dots, l_N)$, then

$$l < l' \quad \text{if} \quad l_N = l'_N,\ l_{N-1} = l'_{N-1},\ \dots\ ,\ l_{i+1} = l'_{i+1},\ l_i > l'_i\,. \tag{133}$$

Thus, for instance, $\tau_j < \tau_1\tau_{j-1} < \dots < \tau_1^{j-2}\tau_2 < \tau_1^j$.

Given this ordering of the basis, it is straightforward to check that the matrix $\overline{H}_N|_{\hat{\mathcal{M}}_k}$ is upper triangular. Its diagonal elements

$$E_{l_1\,l_2\dots l_N} = E_0 - \left(\sum_i i\,l_i\right)\left[2B + a(2N-1) + \sum_i l_i\right] + a\sum_i i^2\,l_i \tag{134}$$

give therefore the energy spectrum of $\overline{H}_N$, and hence of H_N. The spectrum consists of a term

$$E_0 - [2B + a(2N-1)] \sum_i i\, l_i$$

which is constant over $\mathcal{L}_j$, while the remaining two terms split the energies within this subspace, although in general they do not remove all degeneracy. It can be proved with the aid of the normalizability condition that the lowest energy corresponding to each block $\mathcal{L}_j$ is $E_{j0\ldots0}$ while the highest one is $E_{0\ldots1\ldots r}$, where the 1 is in the p-th position, and $j = N\,r + p$ with $1 \le p \le N-1$. This easily implies that the lowest energy in each block $\mathcal{L}_j$ is lower than the lowest energy in the next block $\mathcal{L}_{j+1}$. In particular, the first excited state always lies in $\hat{\mathcal{M}}_1$ (since $\dim \mathcal{M}_0 = 1$), and can thus be exactly computed for any number of particles. On the other hand, some energies in $\mathcal{L}_{j+1}$ might be lower than those of $\mathcal{L}_j$. Note also that the formula (134) for the eigenvalues of the Hamiltonian with potential (79) and (93) is considerably simpler than the one given in Ref. [11].

The spectrum of the three ES potentials (59), (79) and (93) is the same when expressed in terms of B and C_-, although B and C_- are differently related to the parameters appearing in these potentials. Due to the triangular nature of the matrix representing $\overline{H}_N$ in $\hat{\mathcal{M}}_k$, it should not be difficult to give recursive expressions for the eigenfunctions, though we postpone this task for a future work.

4.2 Cases 4–5

The spectrum of the potentials (108) and (127) is highly degenerate, since in this case the energy $E_{l_1\ldots l_N}$ is easily seen to depend only on the single quantum number $j = \sum_{i=1}^N i\, l_i$. More precisely, using the explicit expression (38) for the gauge Hamiltonian and passing to the symmetric variables it is straightforward to show that

$$E_j = E_0 + 2\omega j\,, \qquad j = 0, 1, 2, \ldots\,, \tag{135}$$

where the ground state energy E_0 is given respectively by

$$E_0^{(4)} = N[1 + 2B_- + 2a(N-1)]\,\frac{\omega}{2}\,, \tag{136}$$

$$E_0^{(5)} = N[1 + a(N-1)]\,\omega\,. \tag{137}$$

We have thus rederived (albeit making no use of the underlying root systems) the well-known result that the energy levels in these models are equally spaced, [5].

5 Examples

5.1 ES potential from Case 1

For the ES potential (59)

$$V_N(x) = \sum_k (\eta_0 + \eta_1 \sinh x_k)\,\mathrm{sech}^2 x_k + \frac{g_2}{4}\sum_{j\neq k}\left[\mathrm{csch}^2\left(\frac{x_j - x_k}{2}\right) - \mathrm{sech}^2\left(\frac{x_j + x_k}{2}\right)\right] \tag{138}$$

the gauge Hamiltonian in the symmetric variables is explicitly given by

$$\begin{aligned}\overline{H}_N = &-\sum_{i,j}\left[j\,\tau_i\tau_j + \sum_{k=1}^{j}(j-i-2k)\tau_{i+k}\tau_{j-k}\right]\partial_{\tau_i}\partial_{\tau_j} \\ &-\sum_{i,j}\left[(N-i+1)\tau_{i-1}\tau_{j-1} - \sum_{k=1}^{j-1}(i-j+2k)\tau_{i+k-1}\tau_{j-k-1}\right]\partial_{\tau_i}\partial_{\tau_j} \\ &-\sum_j j\,[2B+1+a(2N-j-1)]\,\tau_j\partial_{\tau_j} - C_-\sum_j (N-j+1)\tau_{j-1}\partial_{\tau_j} \\ &+ a\sum_j (N-j+1)(N-j+2)\tau_{j-2}\,\partial_{\tau_j} + E_0\,,\end{aligned} \tag{139}$$

where B, C_-, and E_0 are given by Eqs. (56)–(57) and (131), respectively. We know from the previous Section that the gauge Hamiltonian (139) preserves each subspace $\hat{\mathcal{M}}_k$ for arbitrary k. For instance, the restriction of $\overline{H}_N$ to $\hat{\mathcal{M}}_2$ has the following matrix with respect to the basis $\{1, \tau_1, \tau_2, \tau_1^2\}$ of $\hat{\mathcal{M}}_2$:

$$\begin{pmatrix} E_0 & -NC_- & aN(N-1) & -2N \\ 0 & E_0 + 1 - \beta & -(N-1)C_- & -2NC_- \\ 0 & 0 & E_0 + 2(a-\beta+1) & 4 \\ 0 & 0 & 0 & E_0 - 2\beta \end{pmatrix}, \tag{140}$$

where we have set

$$\frac{\beta}{2} = a(N-1) + B + 1 < -1\,. \tag{141}$$

(The latter inequality is obtained by imposing the normalizability of the algebraic eigenfunctions belonging to $\hat{\mathcal{M}}_2$.) We can exactly compute four eigenfunctions of the potential (138) with their corresponding energies by diagonalizing the matrix (140). The energies are simply the eigenvalues of this matrix, namely its diagonal elements

$$\begin{aligned} &E_0\,, \\ &E_1 = E_0 - \beta + 1\,, \\ &E_2 = E_0 - 2\beta\,, \\ &E_3 = E_0 + 2(a - \beta + 1)\,. \end{aligned}$$

The corresponding eigenfunctions are given by

$$\psi_i = \mu\,\chi_i\,, \qquad 0 \le i \le 3\,, \tag{142}$$

where

$$\mu = \prod_{j<k}\left(\cosh\frac{x_j+x_k}{2}\,\sinh\frac{x_j-x_k}{2}\right)^a \cdot \prod_k (\operatorname{sech} x_k)^{|B|}\,\mathrm{e}^{\frac{C_-}{2}\,\arctan(\sinh x_k)} \tag{143}$$

and

$$\chi_0 = 1 \tag{144}$$

$$\chi_1 = \tau_1 + \frac{NC_-}{\beta-1} \tag{145}$$

$$\chi_2 = \tau_1^2 - \frac{2\tau_2}{a+1} + \frac{2C_-(Na+1)}{(a+1)(\beta+1)}\,\tau_1 + \frac{N(aN+1)(C_-^2+\beta+1)}{\beta(a+1)(\beta+1)} \tag{146}$$

$$\chi_3 = \tau_2 + \frac{C_-(1-N)}{2a-\beta+1}\tau_1 + \frac{N(N-1)(2a^2-a\beta+a+C_-^2)}{2(a-\beta+1)(2a-\beta+1)}\,, \tag{147}$$

with

$$\tau_1 = \sum_k \sinh x_k\,, \qquad \tau_2 = \sum_{j<k} \sinh x_j\,\sinh x_k\,. \tag{148}$$

Note that, since $\psi_0 = \mu$ has no zeros in the configuration space (47), it is the ground state of the system. We have thus been able to exactly compute some eigenvalues and eigenfunctions of the potential (138) for an *arbitrary* number of particles N.

5.2 QES potential from Case 2

For simplicity, we shall restrict the number of particles to $N=3$ and set $m=1$ to find 4 eigenvalues of the potential (76)

$$V_3(\mathbf{x}) = \sum_{k=1}^{3}\left[B(B-1)\operatorname{csch}^2 x_k + \frac{C_+^2}{4}\sinh^2 x_k + C_+\left(2a+B+\frac{3}{2}\right)\cosh x_k\right] \\ + \frac{a(a-1)}{4}\sum_{\substack{j,k=1\\ j\neq k}}^{3}\left[\operatorname{csch}^2\left(\frac{x_j+x_k}{2}\right) + \operatorname{csch}^2\left(\frac{x_j-x_k}{2}\right)\right], \tag{149}$$

which can be thought of as a deformation of a C_N-type hyperbolic potential. The gauge hamiltonian $\overline{H}_3 = \mu^{-1}\,H_3\,\mu$ can be written as

$$-\sum_{k=1}^{3}(z_k^2-1)\partial_{z_k}^2 - \sum_{k=1}^{3}[(2B+1)z_k + C_+(z_k^2-1)]\,\partial_{z_k} - 2a\sum_{\substack{j,k=1\\ j\neq k}}^{3}\frac{z_k^2-1}{z_k-z_j}\,\partial_{z_k} + C_+\sum_{k=1}^{3} z_k + \epsilon_0\,,$$

where $z_k = \cosh x_k$,

$$\mu = \prod_{1\le j<k\le 3} \left(\sinh \frac{x_j + x_k}{2} \, \sinh \frac{x_j - x_k}{2} \right)^a \cdot \prod_{k=1}^{3} (\sinh x_k)^B \, \mathrm{e}^{\frac{C_+}{2} \cosh x_k} \,, \tag{150}$$

and

$$\epsilon_0 = -\left(3B^2 + 6aB + 5a^2\right) . \tag{151}$$

When expressed in terms of the symmetric variables, the operator $\overline{H}_3$ has been shown in Section 3 to leave invariant the subspace

$$\mathcal{M}_1 = \operatorname{span}\{1, \tau_1, \tau_2, \tau_3\} .$$

The restriction of $\overline{H}_3$ to this subspace is explicitly given by

$$\begin{aligned} \overline{H}_3 \,|_{\mathcal{M}_1} = & - \sum_{j=1}^{3} [2B + 1 + a(5-j)] \, j \, \tau_j \, \partial_{\tau_j} - C_+ \sum_{j=1}^{3} [\tau_1 \tau_j - (j+1)\tau_{j+1}] \, \partial_{\tau_j} \\ & + C_+ \sum_{j=1}^{3} (4-j) \tau_{j-1} \, \partial_{\tau_j} - a \sum_{j=1}^{3} (5-j)(4-j) \tau_{j-2} \, \partial_{\tau_j} + C_+ \tau_1 + \epsilon_0 \, . \end{aligned}$$

The spectral problem for $\overline{H}_3$ in the subspace $\mathcal{M}_1$ thus reduces to diagonalizing the 4×4 matrix of $\overline{H}_3|_{\mathcal{M}_1}$ with respect to the basis $\{1, \tau_1, \tau_2, \tau_3\}$, given by

$$\begin{pmatrix} \epsilon_0 & 3C_+ & -6a & 0 \\ C_+ & \epsilon_0 - \beta - 2a & 2C_+ & -2a \\ 0 & 2C_+ & \epsilon_0 - 2\beta - 2a & C_+ \\ 0 & 0 & 3C_+ & \epsilon_0 - 3\beta \end{pmatrix}$$

with

$$\beta = 2(a + B) + 1 \, .$$

The eigenfunctions of $\overline{H}_3|_{\mathcal{M}_1}$ are

$$\chi_j(\tau) = k_{j0} + k_{j1} \, \tau_1 + k_{j2} \, \tau_2 + \tau_3 \, , \qquad 0 \le j \le 3 \, ,$$

where

$$\begin{aligned} k_{j0} &= \frac{1}{2C_+ \mathcal{E}_j} \left[(\mathcal{E}_j - 2a + 2\beta)(\mathcal{E}_j + 3\beta) - 3C_+^2 \right] , \\ k_{j1} &= \frac{1}{6C_+^2} \left[(\mathcal{E}_j + 2a + 2\beta)(\mathcal{E}_j + 3\beta) - 3C_+^2 \right] , \\ k_{j2} &= \frac{\mathcal{E}_j + 3\beta}{3C_+} \, , \end{aligned}$$

and $\mathcal{E}_j$ is one of the four roots of the characteristic polynomial

$$\mathcal{E}^4 + 2(2a+3\beta)\mathcal{E}^3 + (4a^2 + 18a\beta + 11\beta^2 - 10C_+^2)\mathcal{E}^2 \\ + 6(\beta^3 + 3a\beta^2 + 2a^2\beta + 2aC_+^2 - 5\beta C_+^2)\mathcal{E} + 9C_+^2(-2\beta^2 + 2a\beta + C_+^2)\,.$$

The corresponding eigenfunctions $\psi_j(\mathbf{x}) = \mu(\mathbf{z})\,\chi_j(\tau)$ $(0 \le j \le 3)$ of H_3 have energy $E_j = \mathcal{E}_j + \epsilon_0$. If the parameters a, B and C_+ take the values

$$a = 2\,, \quad B = \frac{3}{2}\,, \quad C_+ = -1\,,$$

compatible with the normalizability of the algebraic eigenfunctions, then the eigenvalues can be calculated numerically:

$$\begin{aligned} E_0 &= -69.7926 \\ E_1 &= -64.1121 \\ E_2 &= -56.4954 \\ E_3 &= -44.5999\,. \end{aligned}$$

The gauge part of the algebraic eigenfunctions is given by

$$\begin{aligned} \chi_0(\tau) &= 0.211595 + 0.376202\,\tau_1 + 0.347522\,\tau_2 + \tau_3 \\ \chi_1(\tau) &= -0.959207 - 0.00694338\,\tau_1 - 1.54596\,\tau_2 + \tau_3 \\ \chi_2(\tau) &= 0.00509379 + 16.3594\,\tau_1 - 4.08486\,\tau_2 + \tau_3 \\ \chi_3(\tau) &= -967.257 + 80.6047\,\tau_1 - 8.05004\,\tau_2 + \tau_3\,. \end{aligned}$$

Since the symmetric variables

$$\begin{aligned} \tau_1 &= \cosh x_1 + \cosh x_2 + \cosh x_3 \\ \tau_2 &= \cosh x_1\, \cosh x_2 + \cosh x_1\, \cosh x_3 + \cosh x_2\, \cosh x_3 \\ \tau_3 &= \cosh x_1\, \cosh x_2\, \cosh x_3 \end{aligned}$$

are all positive, χ_0, and therefore ψ_0, does not vanish in the region (48). This shows that ψ_0 is the ground state of the potential (149).

In the QES cases we are able to compute a few energies and their corresponding eigenfunctions algebraically, but we know nothing about the rest of the spectrum. In fact, it is natural to expect that the potential of this example has some levels between the ones we have obtained.

5.3 ES potential from Case 3

For the ES potential (93)

$$V_N(\mathbf{x}) = \sum_k \left[\frac{C_-^2}{4}\,\mathrm{e}^{-2x_k} + \frac{1}{2}C_0\,C_-\mathrm{e}^{-x_k} + \right] + \frac{g_2}{4}\sum_{j\neq k}\operatorname{csch}^2\left(\frac{x_j - x_k}{2}\right)\,, \tag{152}$$

the gauge Hamiltonian in the symmetric variables is given by

$$
\begin{aligned}
\overline{H}_N = &- \sum_{i,j} \left[j\,\tau_i\tau_j + \sum_{k=1}^{j} (j-i-2k)\tau_{i+k}\tau_{j-k} \right] \partial_{\tau_i}\partial_{\tau_j} \\
&- \sum_j j\,[2B+1+a(2N-j-1)]\,\tau_j\,\partial_{\tau_j} \\
&- C_- \sum_j (N-j+1)\tau_{j-1}\partial_{\tau_j} + E_0\,,
\end{aligned}
\tag{153}
$$

with

$$
B = \frac{1}{2}(C_0+1) \tag{154}
$$

and E_0 given by (131). The gauge Hamiltonian (153) preserves each subspace $\hat{\mathcal{M}}_k$ for arbitrary k. For instance, the restriction of $\overline{H}_N$ to $\hat{\mathcal{M}}_2$ has the following matrix with respect to the basis $\{1, \tau_1, \tau_2, \tau_1^2\}$ of $\hat{\mathcal{M}}_2$:

$$
\begin{pmatrix}
E_0 & -NC_- & 0 & 0 \\
0 & E_0+1-\beta & -(N-1)C_- & -2NC_- \\
0 & 0 & E_0+2(a-\beta+1) & 4 \\
0 & 0 & 0 & E_0-2\beta
\end{pmatrix},
\tag{155}
$$

where β is defined by Eqs. (141) and (154). By diagonalizing this matrix, we can again exactly compute four eigenfunctions with their corresponding energies for the multiparticle potential (152). The energies can be simply read off the diagonal elements of the matrix (155), and coincide with those of the previous example (with B now given by (154)). The corresponding eigenfunctions are given by

$$
\psi_i = \mu\,\chi_i\,, \qquad 0 \le i \le 3\,, \tag{156}
$$

where

$$
\mu = \prod_{j<k} \sinh^a\left(\frac{x_j-x_k}{2}\right) \cdot \exp\left\{ \sum_k \left[\frac{1}{2}\big(C_0+1+a(N-1)\big)\,x_k - \frac{C_-}{2}\mathrm{e}^{-x_k} \right] \right\}, \tag{157}
$$

and

$$
\chi_0 = 1\,, \tag{158}
$$

$$
\chi_1 = \tau_1 + \frac{NC_-}{\beta-1}\,, \tag{159}
$$

$$
\chi_2 = \tau_1^2 - \frac{2\,\tau_2}{a+1} + \frac{2C_-(Na+1)}{(a+1)(\beta+1)}\,\tau_1 + \frac{N(aN+1)C_-^2}{\beta(a+1)(\beta+1)}\,, \tag{160}
$$

$$
\chi_3 = \tau_2 + \frac{C_-(1-N)}{2a-\beta+1}\,\tau_1 + \frac{N(N-1)C_-^2}{2(a-\beta+1)(2a-\beta+1)}\,, \tag{161}
$$

with

$$\tau_1 = \sum_k e^{x_k}\,, \qquad \tau_2 = \sum_{j<k} e^{x_j+x_k}\,. \tag{162}$$

Again, $\psi_0 = \mu$ has no zeros in the configuration space (47), and is therefore the ground state of the system. As in the previous example, we have exactly computed some eigenvalues and eigenfunctions of the potential (152) for an arbitrary number of particles.

5.4 QES potential from Case 3

As a final example, consider the QES potential (90) with $m = 1$

$$\begin{aligned} V_N(\mathbf{x}) = &\sum_k \left[\frac{C_-^2}{4}\,\mathrm{e}^{-2x_k} + \frac{1}{2}C_0\,C_-\mathrm{e}^{-x_k} + C_+\big(a(N-1) + \frac{C_0}{2} + 2\big)\mathrm{e}^{x_k} + \frac{C_+^2}{4}\,\mathrm{e}^{2x_k}\right] \\ &+ \frac{g_2}{4}\sum_{j\neq k}\operatorname{csch}^2\left(\frac{x_j - x_k}{2}\right). \end{aligned} \tag{163}$$

The restriction of the gauge Hamiltonian $\overline{H}_N$ to the subspace

$$\mathcal{M}_1 = \operatorname{span}\{1, \tau_1, \dots, \tau_N\}$$

is given in terms of the symmetric variables τ_j by the following expression

$$\begin{aligned} \overline{H}_N|_{\mathcal{M}_1} = \epsilon_0 &+ C_+\,\tau_1\left(1 - \sum_j \tau_j\,\partial_{\tau_j}\right) + C_+\sum_j (j+1)\,\tau_{j+1}\partial_{\tau_j} \\ &- \sum_j\,[C_0 + a(2N-j-1) + 2]\,j\,\tau_j\,\partial_{\tau_j} - C_-\sum_j (N-j+1)\,\tau_{j-1}\,\partial_{\tau_j}\,, \end{aligned} \tag{164}$$

with

$$\epsilon_0 = -\frac{N}{4}\left[\big(C_0 + a\,(N-1) + 1\big)^2 + \frac{a^2}{3}\,(N^2-1) + 2\,C_-C_+\right]. \tag{165}$$

The matrix $(h_{ij})_{0\le i,j\le N}$ of $\overline{H}_N|_{\mathcal{M}_1}$ with respect to the basis $\{1 \equiv \tau_0, \tau_1, \dots, \tau_N\}$ of $\mathcal{M}_1$ is therefore tridiagonal, with nontrivial elements

$$h_{j,j-1} = j\,C_+\,, \qquad h_{j,j+1} = -C_-(N-j)\,, \tag{166}$$

$$h_{jj} = \epsilon_0 - j\,[C_0 + a(2N-j-1) + 2] \equiv h_j\,. \tag{167}$$

The tridiagonal character of the matrix (h_{ij}) has important consequences for the calculation of the eigenfunctions of $\overline{H}_N$ lying in $\mathcal{M}_1$. Indeed, if we write one such eigenfunction with eigenvalue E as

$$\chi_E(\tau) = \sum_{j=0}^{N} \tilde{\pi}_j(E)\,\tau_j\,, \tag{168}$$

it follows from Eqs. (166)–(167) that the coefficients $\tilde{\pi}_j(E)$ satisfy the following relations

$$jC_+\,\tilde{\pi}_{j-1}(E) + (h_j - E)\tilde{\pi}_j(E) - C_-(N-j)\tilde{\pi}_{j+1}(E) = 0\,, \qquad j = 0,1,2,\dots,N\,, \quad (169)$$

where we can take $\tilde{\pi}_{-1} = 0$ and, without loss of generality, $\tilde{\pi}_0 = 1$. Let us now regard the energy E in Eq. (169) as a real variable, and let us consider the recurrence relation (169) for arbitrary values of $j \in \mathbf{N}$. Following the usual procedure, we introduce new normalized coefficients $\pi_j(E)$ through the formula

$$\pi_j(E) = \frac{N!}{(N-j)!}\,(-C_-)^j\,\tilde{\pi}_j(E)\,. \qquad (170)$$

Note that $\tilde{\pi}_0(E) = 1$ implies that $\pi_0(E) = 1$. The coefficients $\pi_j(E)$ then satisfy the following three-term recurrence relation in canonical form

$$\pi_{j+1}(E) = (E - h_j)\,\pi_j(E) - \alpha_j\,\pi_{j-1}(E)\,, \qquad (171)$$

with

$$\alpha_j = -j(N-j+1)C_-C_+\,. \qquad (172)$$

The functions $\pi_j(E)$ defined by the latter relation with the initial conditions $\pi_{-1}(E) = 0$, $\pi_0(E) = 1$ are monic polynomials of degree j. It is well known that the canonical form of the recurrence relation (172) entails that the polynomials $\pi_j(E)$ are an orthogonal family with respect to a suitably defined weight functional, [17]. Furthermore, the fact that the coefficient α_j vanishes for $j = N+1$ implies that this functional is not positive definite; in fact, the polynomials π_j with $j \geq N+1$ must have zero norm. Therefore, the polynomial family $\{\pi_j(E)\}_{j\geq 0}$ is *weakly* orthogonal, cf. [17]. We have thus associated in a natural way a weakly orthogonal polynomial family to the QES many-body potential (163). Recall, in this respect, that it is possible to construct a weakly orthogonal polynomial family for (almost) every QES one-particle potential on the line, [18].

Going back to the calculation of the eigenfunctions of $\overline{H}_N$ belonging to $\mathcal{M}_1$, note first of all that their energies are the zeros of the *critical polynomial* $\pi_{N+1}(E)$. This can be seen, for instance, by observing that the j-th principal minor $\delta_j(E)$ of the matrix $E-(h_{ij})$ satisfies the same recurrence relation as $\pi_j(E)$, with the same initial conditions, so that

$$\delta_j(E) = \pi_j(E)\,, \qquad (173)$$

and, in particular,

$$\delta_{N+1}(E) = \det\big(E - (h_{ij})\big) = \pi_{N+1}(E)\,. \qquad (174)$$

Since $C_+C_- < 0$ for the potential (163), it follows from (172) that α_j is positive for $j = 1,2,\dots,N$. This implies, [19], that the critical polynomial $\pi_{N+1}(E)$ has $N+1$ distinct real roots $E_0 < E_1 < \cdots < E_N$. By the previous equation, the spectrum of the restriction of $\overline{H}_N$ to $\mathcal{M}_1$ consists of the $N+1$ real eigenvalues $E_0 < \cdots < E_N$. The respective (unnormalized) eigenfunctions $\chi_i \equiv \chi_{E_i}$ are given by

$$\chi_i(\tau) = 1 + \frac{1}{N!} \sum_{j=1}^{N} (-1)^j \, \frac{(N-j)!}{C_-^j} \, \pi_j(E_i) \, \tau_j \,, \tag{175}$$

where the coefficients $\pi_j(E_i)$ are calculated either from the recurrence relation (171) (for $E = E_i$), or by computing the minors $\delta_j(E_i)$. The corresponding eigenfunctions of the physical Hamiltonian H_N are obtained by multiplying each function χ_i in (175) by the gauge factor

$$\mu = \prod_{j<k} \sinh^a \left(\frac{x_j - x_k}{2} \right) \cdot \exp \left\{ \frac{1}{2} \sum_k \left[\big(C_0 + a(N-1) + 1 \big) x_k + C_+ \mathrm{e}^{x_k} - C_- \mathrm{e}^{-x_k} \right] \right\} . \tag{176}$$

We can also express the eigenvalues of $\overline{H}_N|_{\mathcal{M}_1}$ and their eigenfunctions using continued fractions, by working with the quotients

$$q_j(E) = \frac{\pi_j(E)}{\pi_{j-1}(E)} \,, \qquad j = 1, 2, \dots \,. \tag{177}$$

Then Eq. (171) becomes

$$q_{j+1}(E) = E - h_j - \frac{\alpha_j}{q_j(E)} \,, \tag{178}$$

from which we immediately obtain the continued fraction expansion

$$q_{j+1}(E) = E - h_j - \frac{\alpha_j}{E - h_{j-1} -} \, \frac{\alpha_{j-1}}{E - h_{j-2} -} \, \cdots \, \frac{\alpha_2}{E - h_1 -} \, \frac{\alpha_1}{E - h_0} \,. \tag{179}$$

The eigenvalue equation in this formalism can be obtained by imposing the vanishing of $q_{N+1}(E)$ (since it is proportional to $\pi_{N+1}(E)$, by Eq. (177)), that is

$$E - h_N = \frac{\alpha_N}{E - h_{N-1} -} \, \frac{\alpha_{N-1}}{E - h_{N-2} -} \, \cdots \, \frac{\alpha_2}{E - h_1 -} \, \frac{\alpha_1}{E - h_0} \,. \tag{180}$$

Alternatively, solving Eq. (178) for q_j we obtain

$$q_j(E) = \frac{\alpha_j}{E - h_j - q_{j+1}(E)} \,, \tag{181}$$

and an equivalent form of the eigenvalue equation (180) follows by expressing $q_1(E) = E - h_0$ as a continued fraction:

$$E - h_0 = \frac{\alpha_1}{E - h_1 -} \, \frac{\alpha_2}{E - h_2 -} \, \cdots \, \frac{\alpha_{N-1}}{E - h_{N-1} -} \, \frac{\alpha_N}{E - h_N} \tag{182}$$

(since $q_{N+1}(E)$ vanishes if and only if E is an eigenvalue). The eigenfunctions are still given by (175), where of course now

$$\pi_j(E) = \prod_{i=1}^{j} q_i(E)\,. \tag{183}$$

Consider, as an example, the potential (163) with $N = 3$ particles, for which

$$\epsilon_0 = -\frac{3}{4}(C_0 + 2a + 1)^2 - 2a^2 - \frac{3}{2}C_-C_+\,. \tag{184}$$

The eigenfunctions of $\overline{H}_3|_{\mathcal{M}_1}$ can be expressed as

$$\chi_j(\tau) = 1 + k_{j1}\,\tau_1 + k_{j2}\,\tau_2 + k_{j3}\,\tau_3\,, \tag{185}$$

where the coefficients $k_{jl} = \tilde{\pi}_l(E_j)$ are given by

$$k_{j1} = -\frac{\mathcal{E}_j}{3C_-}\,, \tag{186}$$

$$k_{j2} = \frac{1}{6C_-^2}\left[\mathcal{E}_j^2 + (2a+\beta)\mathcal{E}_j + 3C_-C_+\right], \tag{187}$$

$$k_{j3} = -\frac{1}{6C_-^3}\left[\mathcal{E}_j^3 + (4a+3\beta)\mathcal{E}_j^2 + (4a^2+6a\beta+2\beta^2+7C_-C_+)\mathcal{E}_j + 6(a+\beta)C_-C_+\right]. \tag{188}$$

In the latter equations we have set

$$\beta = 2a + C_0 + 2\,, \tag{189}$$

and $\mathcal{E}_j = E_j - \epsilon_0$ is one of the roots of the polynomial

$$\begin{aligned}\pi_4(\mathcal{E}+\epsilon_0) &= \mathcal{E}^4 + (4a+6\beta)\mathcal{E}^3 + (4a^2+18a\beta+11\beta^2+10C_-C_+)\,\mathcal{E}^2\\ &\quad + 6\big(\beta(2a^2+3a\beta+\beta^2) + (2a+5\beta)C_-C_+\big)\,\mathcal{E} + 9C_-C_+\big(2\beta(a+\beta) + C_-C_+\big)\,.\end{aligned} \tag{190}$$

For instance, if the parameters in the potential are given by

$$a = 2\,,\qquad C_- = 2\,,\qquad C_0 = 1\,,\qquad C_+ = -1 \tag{191}$$

then

$$\epsilon_0 = -32$$

and

$$\begin{aligned}E_0 &= -54.5584\\ E_1 &= -49.5497\\ E_2 &= -42.4323\\ E_3 &= -31.4596\,.\end{aligned}$$

The eigenfunctions of $\overline{H}_3$ in $\mathcal{M}_1$ are

$$\begin{aligned}
\chi_0(\tau) &= 1 + 3.75974\,\tau_1 + 10.6142\tau_2 + 20.4323\,\tau_3\\
\chi_1(\tau) &= 1 + 2.92495\,\tau_1 + 4.5394\,\tau_2 - 3.94697\,\tau_3\\
\chi_2(\tau) &= 1 + 1.73871\,\tau_1 - 0.496778\,\tau_2 + 0.141027\,\tau_3\\
\chi_3(\tau) &= 1 - 0.090071\,\tau_1 + 0.00986437\,\tau_2 - 0.00137384\,\tau_3\,.
\end{aligned}$$

Since the symmetric variables, given in this case by (162) and

$$\tau_3 = \sum_{i<j<k} \mathrm{e}^{x_i+x_j+x_k}\,, \tag{192}$$

are all positive, the function $\psi_0 = \mu\chi_0$ is once again the ground state of the Hamiltonian H_3.

6 Summary and Conclusions

We have used Calogero's construction of classical solvable many-body systems and applied it to the most general one-dimensional quasi-exactly solvable normalizable Schrödinger operator in the line. The corresponding quantum many-body Hamiltonians have been shown to have an algebraic structure that enables us to compute part or all of their spectrum by straightforward algebraic means. In all cases, if the one-dimensional *seed potential* is ES (QES) then the corresponding many-body potential constructed on it is also ES (QES).

The QES examples include a generalization of the sextic deformation of Olshanetsky–Perelomov's B_N rational model recently found in [10], as well as some new deformations of the standard hyperbolic BC_N potential. We have also found an additonal QES deformation of the B_N rational model not related to the one considered in Ref. [10].

The ES cases include A_N and B_N rational models with harmonic force and some hyperbolic OP potentials of BC_N type, as well as the A_N hyperbolic model with an external Morse potential discussed by Inozemtsev and Meshcheryakov, [11]. Since we have only looked for Hamiltonians having normalizable eigenfunctions, the models treated in [5] with no discrete spectrum do not appear in this work. The fact that we have considered only non-periodic one-dimensional seed potentials, [20], explains why the trigonometric integrable Hamiltonians such as Sutherland's model are also absent. In a future work we plan to perform a similar analysis using *periodic* seed potentials which could lead to new results on the elliptic case, where much less is known.

The ES problems have the additional feature that their associated gauge Hamiltonian $\overline{H}_N$ preserves not only the subspaces $\mathcal{M}_k$ but the smaller subspaces $\hat{\mathcal{M}}_k$ defined by (34). This implies that the number of particles N represents no difficulty in these cases, and allows us to calculate some levels and eigenfunctions for arbitrary N. Moreover, we have shown that an ordered basis in each subspace $\hat{\mathcal{M}}_k$ can be chosen so that the action of $\overline{H}_N$ is triangular. As a consequence, the diagonal terms automatically give the discrete spectrum, and recursive expressions for the eigenfunctions can in principle be written.

It should be stressed that the results we have obtained depend greatly on the ansatz (37) for the gauge factor μ. Our ansatz has been chosen by analogy with the one-dimensional case, but it is clear that more general ansätze could lead to more general results. In his recent work [21], Calogero investigates a very general ansatz which gives rise to higher-body interactions. A systematic analysis of this method should focus on the equivalence problem, attempting to solve it in full generality.

7 Acknowledgments

It is our pleasure to thank F. Calogero, A. Perelomov, and O. Ragnisco for useful discussions. D. G.-U. is also glad to acknowledge F. Calogero's warm hospitality at Universitá di Roma (La Sapienza) while this work was in progress.

8 Appendix

The following expressions are useful in the transition from the canonical coordinates $\mathbf{z}$ to the symmetric variables τ. (All indices run from 1 to N, with the restrictions indicated under the summation symbol.)

$$\Lambda_1^{(p)} \equiv \sum_{k\neq j} \frac{z_k^p}{z_k - z_j}$$

$$\Lambda_1^{(0)} = 0 \tag{193}$$

$$\Lambda_1^{(1)} = \frac{1}{2}N(N-1) \tag{194}$$

$$\Lambda_1^{(2)} = (N-1)\tau_1 \tag{195}$$

$$\Lambda_2^{(p)} \equiv \sum_{j\neq k\neq l\neq j} \frac{z_k^p}{(z_k - z_j)(z_k - z_l)}$$

$$\Lambda_2^{(0)} = \Lambda_2^{(1)} = 0 \tag{196}$$

$$\Lambda_2^{(2)} = \frac{1}{3}N(N-1)(N-2) \tag{197}$$

The following expressions are useful to express differential operators in terms of the symmetric variables (it is understood that $\tau_k = 0$ for $k > N+1$):

$$\sum_j z_j^p\,\partial_{z_j} = \sum_j B_j^{(p)}\,\partial_{\tau_j}$$

$$B_j^{(0)} = (N-j+1)\tau_{j-1} \tag{198}$$
$$B_j^{(1)} = j\tau_j \tag{199}$$
$$B_j^{(2)} = \tau_1\tau_j - (j+1)\tau_{j+1} \tag{200}$$

$$\sum_k z_k^p\, \partial_{z_k}^2 = \sum_{i,j} A_{ij}^{(p)}\, \partial_{\tau_i}\partial_{\tau_j}$$

$$A_{ij}^{(0)} = (N-i+1)\tau_{i-1}\tau_{j-1} - \sum_{k=1}^{j-1}(i-j+2k)\tau_{i+k-1}\tau_{j-k-1} \tag{201}$$
$$A_{ij}^{(1)} = \sum_{k=0}^{j-1}(i-j+2k+1)\tau_{i+k}\tau_{j-k-1} \tag{202}$$
$$A_{ij}^{(2)} = j\tau_i\tau_j + \sum_{k=1}^{j}(j-i-2k)\tau_{i+k}\tau_{j-k} \tag{203}$$

$$2\sum_{j\neq k}\frac{z_k^p}{z_k-z_j}\,\partial_{z_k} = \sum_j C_j^{(p)}\,\partial_{\tau_j}$$

$$C_j^{(0)} = -(N-j+1)(N-j+2)\tau_{j-2} \tag{204}$$
$$C_j^{(1)} = (N-j)(N-j+1)\tau_{j-1} \tag{205}$$
$$C_j^{(2)} = j(2N-j-1)\tau_j \tag{206}$$

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